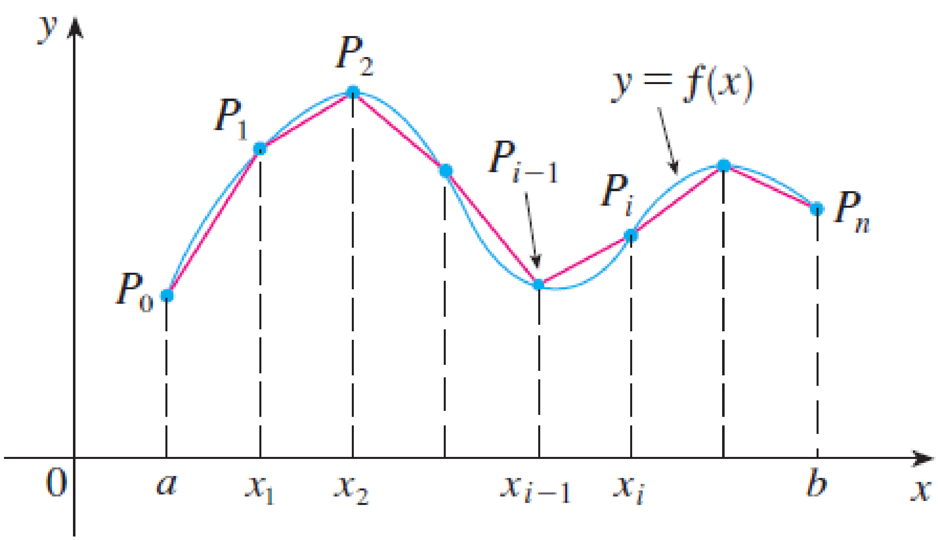
Closing Wed: HW_7B, 7C (8.1)

8.1 Arc Length

Goal: Given y = f(x) from x = a to x = b.
Find the ***length*** along the curve.



Derivation:

1. Break into *n* subdivision:

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x$$

2. Compute $y_i = f(x_i)$.

3. Compute the straight line distance from (x_i, y_i) to (x_{i+1}, y_{i+1}).

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$
$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$
$$= \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y_i)^2}{(\Delta x)^2}\right)}$$
$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

4. Add these distances up.

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Examples:

1. Find the arc length of

$$y = \ln(\sec(x))$$

from x = 0 to x = π/4.

2. Set up the integral that gives the arc length of $y = x^4$ on x = 0 to x = 4.

3. Find the arc length of

$$y = \frac{x^3}{3} + \frac{1}{4x}$$

from x = 1 to x = 2.

4. Find the arc length of $y = 4x^{3/2}$ from x = 0 to x = 3.

Side Note: The Arc Length function

In 2D and 3D motion problems, the arch length plays an important role. In those settings you have parametric equations for motion:

x = x(u) , y = y(u)

and so

$$\text{Length} = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int \sqrt{1+\left(\frac{y'(u)}{x'(u)}\right)^2}\,x'(u)du$$

$$= \int \sqrt{\left(x'(u)\right)^2+\left(y'(u)\right)^2}\,du$$

Thus, the distance traveled by the particle from time u = 0 to u = t is given by

$$s(t) = \int_0^t \sqrt{\left(x'(u)\right)^2+\left(y'(u)\right)^2}\,du$$

This is called the *arc length function*.

A important note is that the speed of the object is:

$$s'(t) = \sqrt{\left(x'(u)\right)^2+\left(y'(u)\right)^2}$$

Example:

1. $x(u) = 3u$, $y(u) = 4u$.
   Find the arc length function.

2. $x(u) = u$, $y(u) = 4u^{3/2}$
   Find the arc length function.